Tail Spectral Density Estimation and Its Uncertainty Quantification: Another Look at Tail Dependent Time Series Analysis

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Presentation Overview

Introduction

Examples of extreme values where tail dependence may lay:

- Huge stock drops
- Extreme weather conditions
- Large credit defaults

- Independence assumption typically does not hold in regard to time series data.
- Besides, Random variables can show different patterns of dependence in non-tail and extreme deviations, ignoring which may yield misleading conclusions.
- We would like to make contribution to describing dependence in the tail part of a distribution.

Introduction

- Traditional mean-based dependence measured by correlation

$$\text{cor}(X, Y) = \frac{\mathrm{E}[\{X - \mathrm{E}(X)\}\{Y - \mathrm{E}(Y)\}]}{\sqrt{\text{var}(x)\text{var}(Y)}}$$

  concerns the co-movement around mean,

$$e.g. X > \mathrm{E}(X) \longleftrightarrow Y > \mathrm{E}(Y).$$

- Tail dependence concerns the co-occurance of extremal events,

$$e.g., X \text{ being extremal} \longleftrightarrow Y \text{ being extremal},$$

  and was firstly introduced as the dependence in the bivariate distribution function.

$$\lambda_{up} = \lim_{u \to 1^-} Pr(X_2 > F_{X_2}^{-1}(u) | X_1 > F_{X_1}^{-1}(u)),$$

  where λ_{up} is tail dependence of the bivariate upper tails.

Tail Autocorrelation

- In the time series setting, tail dependence is often measured at different lags. Zhang(2006) defines a k-lag dependence for time series data: let $\mathcal{U}_F = \lim_{u \uparrow 1} F^{-1}(u)$

$$\rho_k = \lim_{x \uparrow \mathcal{U}_F} \mathrm{pr}(X_{k+1} > x \mid X_1 > x).$$

- As a result, similar to the traditional autocorrelation, there can be an infinite number of tail dependence measurements, one at each lag.
- Traditional autocorrelation analysis, however, can not be used to study tail dependence. Instead, consider the tail autocorrelation

$$\rho_{k,n} = \frac{\mathrm{pr}(X_{k+1} > x_n \mid X_1 > x_n) - \mathrm{pr}(X_{k+1} > x_n)}{1 - \mathrm{pr}(X_1 > x_n)},$$

which could be viewed as a standardized pre-asymptotic version of the lag-k tail dependence.

Tail Spectral Density Estimation

- Let $\iota = \sqrt{-1}$, we want to get a good estimation of the tail spectral density :

$$f_n(\lambda) = \frac{1}{2\pi} \sum_{|k|<n} \rho_{k,n} e^{\iota k \lambda},$$

  and naturally extend the conventional spectral density to the tail setting using the tail autocorrelations.
- We use the lag-window estimator, which has been widely implemented with non-tail setting :

$$\hat{f}_n(\lambda) = \frac{1}{2\pi} \sum_{|k|<n} \hat{\rho}_{k,n} e^{\iota k \lambda} K\left(\frac{k}{B_n}\right),$$

  where K is a kernel function, and B_n is a positive bandwidth sequence.

Introduction

- Consider a stationary system $X_i = G(\mathcal{F}_i) = (\cdots, \epsilon_{i-1}, \epsilon_i)$, where ϵ_j, are iid innovations. And let $\epsilon_0^\star$ be an innovation that has the same distribution as ϵ_0 but independent of $(\epsilon_k)_{k\in\mathbb{R}}$, then $X_i^\star = G(\mathcal{F}_i) = (\cdots, \epsilon_{-1}, \epsilon_0^\star, \epsilon_1, \cdots, \epsilon_{i-1}, \epsilon_i)$.
- T.Zhang(2021) proposed to consider adversarial tail dependence measure

$$\theta_x(i) = \sup_{z\geq x} \mathrm{pr}(X_i^\star \leq z \mid X_i \geq z).$$

- We say a time series process (X_i) is tail adversarial $q-$stable, or $(X_i) \in \mathrm{TAS}_q$ if

$$\lim_{x\uparrow\mathcal{U}_F} \Theta_{x,q}(0) = \lim_{x\uparrow\mathcal{U}_F} \sum_{i=0}^{\infty} \{\theta_x(i)\}^{1/q} < \infty,$$

for $q \geq 1$.

Tail Adversarial Stability Framework

- We say a time series process (X_i) is geometrically tail adversarial stable, or $(X_i) \in$ GTAS if there exists constant $c^\star \in (0, \infty)$,and $\phi \in (0, 1)$ such that
$$\theta_X(i) \leq c^\star \phi^i, i \geq 0,$$
holds for some x that is close enough to $\mathcal{U}_F$.
- TAS framework lays a convenient and mathematically rigorous foundation for developing limit theorems of tail dependent time series.
- It's been proven that TAS could produce the same results under weaker conditions in some scenarios.
- It also provides flexibility and interpretability to data analysis practice.

Main Results

- Recall tail spectral density estimator :

$$\hat{f}_n(\lambda) = \frac{1}{2\pi} \sum_{|k|<n} \hat{\rho}_{k,n} e^{\iota k\lambda} K\left(\frac{k}{B_n}\right).$$

Theorem (Consistency)

Under some regularity conditions including $(X_i) \in TAS_4$, for any $\lambda \in [0, 2\pi)$,

$$\hat{f}_n(\lambda) - f_n(\lambda) \to_p 0.$$

- This asymptotic consistency result requires weaker conditions on tail dependence and allows more extremal tails as $n\bar{F}(x_n) \to \infty$ v.s. $n^{1/3}\bar{F}(x_n) \to \infty$.

Main Results

Theorem (Central Limit Theorem)

Under some regularity conditions including $(X_i) \in GTAS$ *then*

(i) for $\lambda \in [0, 2\pi) \setminus \{0, \pi\}$*,where* $f_n(\lambda)$ *is bounded away from zero for all large* n*,*

$$\{\kappa^{1/2} f_n(\lambda)\}^{-1}(B_n^{-1} n)^{1/2}\{\hat{f}_n(\lambda) - f_n(\lambda)\} \to_d N(0, 1);$$

and (ii) for $\lambda \in \{0, \pi\}$ *where* $f_n(\lambda)$ *is bounded away from zero for all large* n*,*

$$\{\kappa^{1/2} f_n(\lambda)\}^{-1}(B_n^{-1} n)^{1/2}\{\hat{f}_n(\lambda) - f_n(\lambda)\} \to_d N(0, 2),$$

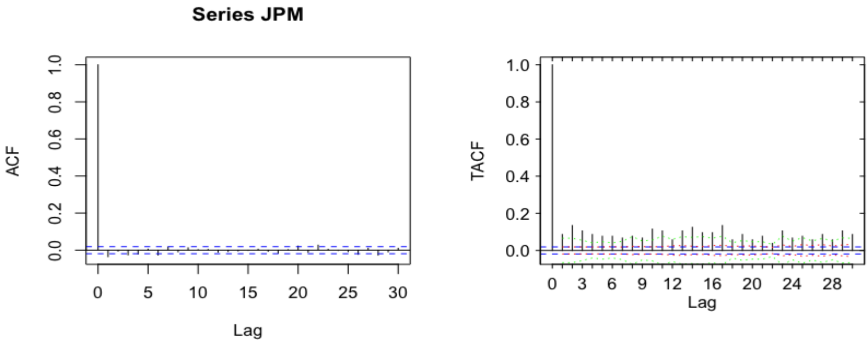
where $\kappa = \int_{u \in \mathcal{R}} K^2 du$.

- Fun fact: different variance!

Data Implementation

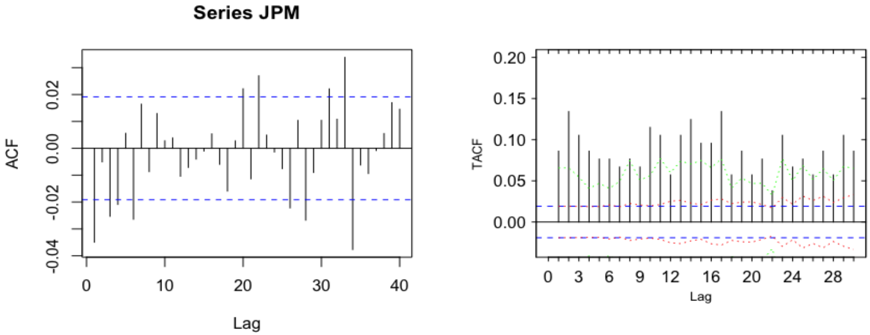
- Time series data in the fields of climate science, ecological science, finance, economics, and so on could contain some dependency in their tail part.
- We present the data application results of a financial data set and a temperature data set.
- And compared the results of traditional spectral density estimation and tail spectral density analysis.

A Financial Data Set



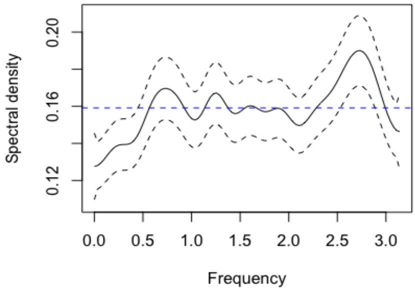
- Data set : JPM daily stock price 03/17/1980 to 10/15/2021. Lower tail part of the log return. 99% quantile as threshold.
- Autocorrelation function for conventional method(left) and tail spectral density function(right) with noninformative lag 0.

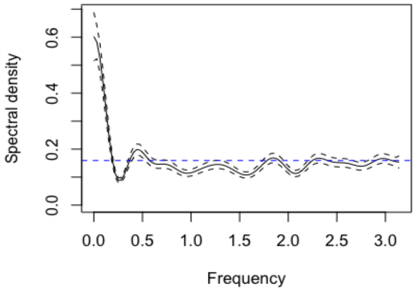
A Financial Data Set



- Autocorrelation function for conventional method(left) and tail spectral density function(right).

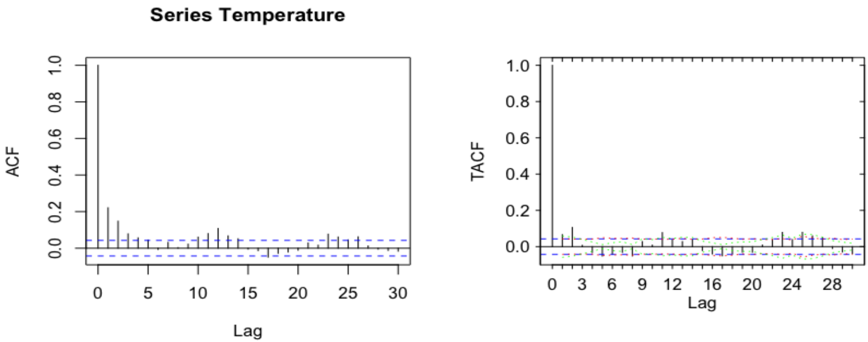
A Financial Data Set





- Spectral Density Plots: Traditional Spectral Density Analysis (left), and Tail Spectral Density Method (right).

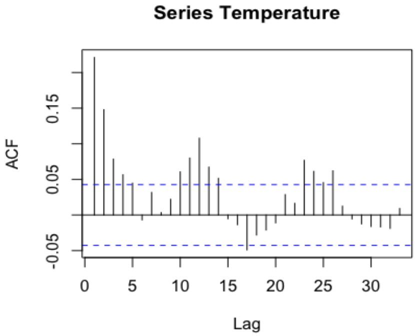
A Temperature Data Set

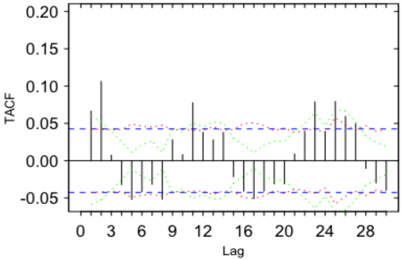


- Data set : Monthly averages of daily high temperatures in the United States, 03/1840 to 05/2016. 95% quantile as threshold.
- Autocorrelation function for traditional method(left) and tail spectral density function(right) with noninformative lag 0.

Data Implementation

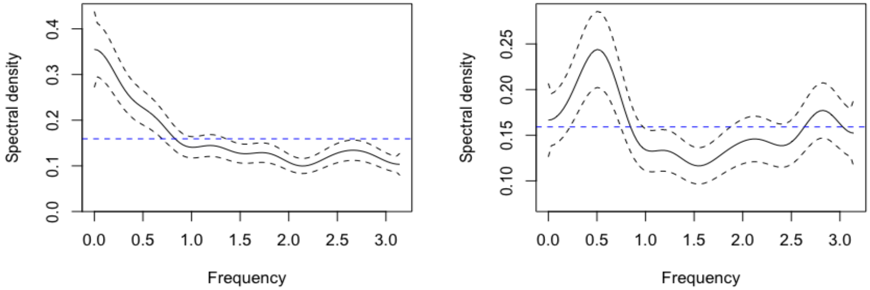
A Temperature Data Set





- Autocorrelation function for traditional method(left) and tail spectral density function(right).

A Temperature Data Set



- Spectral Density Plots: Traditional Spectral Density Analysis (left), and Tail Spectral Density Method (right).

Conclusion

Takeaway

- We considered the estimation for tail spectral density with focus on serial dependence in the tail region.
- TAS framework could achieve a weaker condition under certain circumstances.
- The resulted method constructs confidence interval that gauges the statistical uncertainty of the estimator, and captures the tail dependency traditional method otherwise cannot.

Reference

- Zhang, T., & Xu, B. (2023). Tail Spectral Density Estimation and Its Uncertainty Quantification: Another Look at Tail Dependent Time Series Analysis. *Journal of the American Statistical Association*, forthcoming.

Many thanks for your time!